PLASMA DIFFUSION IN A TOROIDAL STELLARATOR

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 9, No. 6, pp. 3–12, 1968

Particle trajectories in a gently toroidal stellarator are investigated. The distribution function for the particles is determined in the absence of collisions.

The lines of force of a magnetic field form surfaces in magnetic traps with rotational transformation. As a rule the drift trajectories of plasma particles in such traps form surfaces which are fairly close to the magnetic surface. However, the drift surfaces of particles moving with small velocities along the lines of force may suffer fairly strong departures from the magnetic surfaces. The mixing which arises as a result of this can lead to a considerable increase of particle and heat fluxes. These fluxes were found in [1] for the case of toroidal systems with axial symmetry. A toroidal stellarator does not possess axial symmetry. The particle orbits, which perform a fairly complicated precession in such traps, can depart even further from the magnetic surfaces [2–4]. This is most likely for particles with small longitudinal velocities, particularly particles trapped in their motion along the lines of force within regions where the magnetic field is a minimum. Such particle motion presents an exceedingly complicated picture since the particles are situated in a field which is the superposition of two potential wells: the first associated with the fact that the field is toroidal (as in the axiasymmetric case) and the second arising from the fact that the magnetic field is helical. For this reason not even an estimate has been made up to the present of the mixing which results.

In what follows we consider the limiting case of a gently toroidal stellarator, for which the drift equations of particle motion are integrable. The particle distribution function in the absence of collisions is automatically obtained at the same time. The effect of collisions is subsequently calculated from perturbation theory.

1. **Particle motion in a stellarator with a strong helical magnetic field.** We consider particle motion in a helical magnetic field which, to be specific, is taken to have three strands. The field is "bent" into a torus of large radius. Close to the axis of the helical field the following additional approximations can be made to simplify the problem:

a) cross sections of the magnetic surfaces are concentric circles;

b) the contribution of transverse components of the helical magnetic field to the diamagnetic particle drift are negligibly small compared with the contribution arising from the inhomogeniety of the longitudinal field B_z.

Close to the magnetic axis the following expression can be used [5]:

$$B_z = B_0 [1 - \varepsilon_h \cos 3(\vartheta - \alpha z) - \varepsilon_t \cos \vartheta] \qquad \left(\varepsilon_h = \frac{27b}{16B_0}(\alpha r)^3,\ \alpha = \frac{2\pi}{L}\right). \tag{1.1}$$

Here α is the pitch of the helical field; the last term on the right-hand side represents the toroidal correction, so that

$$\varepsilon_t \ll \varepsilon_h \tag{1.2}$$

The equations of motion of a charged particle with energy E and adiabatic invariant μ can be represented in the drift approximation in the following form (Φ (r) is the electric potential):

$$r\frac{d(\vartheta - \alpha z)}{dt} = -\alpha r\left[v_\parallel - \frac{c}{\alpha r B_0}\frac{d\Phi}{dr}\right] - \frac{\mu B_0}{m_j \omega_{cj} r}(3\varepsilon_h \cos 3(\vartheta - \alpha r) + \varepsilon_t \cos \vartheta)\,; \tag{1.3}$$

$$\frac{dr}{dt} = -\frac{\mu B_0}{m_j \omega_{cj} r}(3\varepsilon_h \sin 3(\vartheta - \alpha z) + \varepsilon_t \sin \vartheta)\,; \tag{1.4}$$

$$\frac{dz}{dt} = v_\parallel,\ v_\parallel = \pm\left(\frac{2}{m_j}[E - e_j\Phi - \mu B_0(1 - \varepsilon_h \cos 3(\vartheta - \alpha z) - \varepsilon_t \cos \vartheta)]\right)^{1/2}. \tag{1.5}$$

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The term $(B_\vartheta/B_0)\, v_\parallel$ is omitted in Eq. (1.3); it is assumed that it is small compared with the first term on the right-hand side; this is valid close to the axis*. Centrifugal drift is not allowed for in Eq. (1.3) and (1.4) since we are dealing with trapped particles (having small $v_\parallel$ components). It follows from [6] that this also imposes an upper limit on the electric field strength.

By means of the substitution $\varphi = 3(\vartheta - \alpha z)$, the system of equations (1.3)–(1.5) in the zeroth approximation ($\varepsilon_t = 0$) can be reduced to a form similar to the equations of motion of particles in an axisymmetric torus, and if we neglect the departure of the particles Δr from the magnetic surface (Δr is in fact a small quantity proportional to the Larmor radius), then instead of (1.3) and (1.5) we can write the following equation to describe the motion with respect to the φ coordinate:

$$r_0 \frac{d\varphi}{dt} = -3\alpha r_0 v \sqrt{\varepsilon_h [2\varkappa^2 - 1 + \cos\varphi]}, \quad v = \left(\frac{2\mu B_0}{m_j}\right)^{1/2},$$

$$\varkappa^2 = \frac{1}{2v^2\varepsilon_h}\left[\left(\frac{2}{m_j}[E - e\Phi - \mu B_0(1-\varepsilon_h)]\right)^{1/2} - \frac{c}{\alpha B_0 r}\frac{d\Phi}{dr}\right]^2. \tag{1.6}$$

It follows from this that as in the case of an axisymmetric torus [6, 7], the motion of trapped particles can be described in terms of elliptic functions with modulus

$$\varkappa^2 \leqslant 1. \tag{1.7}$$

The oscillation period of the trapped particles is equal to

$$\tau = \frac{4}{3\alpha v\sqrt{2\varepsilon_h}} \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{\varkappa^2 - \sin^2 {}^1\!/_2\varphi}} = \frac{4\sqrt{2}}{3\alpha v\sqrt{\varepsilon_h}} K(\varkappa), \tag{1.8}$$

where $K(\varkappa)$ is a complete elliptic integral of the first kind, φ_0 is a zero of the expression in the radicand. The corresponding orbits in the $\vartheta - \alpha z$, r plane are shown in Fig. 1 (we will call these "bananas"). The trapped particles move in the angular intervals

$$^2/_3\pi(l - {}^1/_2) \leqslant \vartheta - \alpha z \leqslant {}^2/_3\pi(l + {}^1/_2) \quad (l = 0, 1, 2, \ldots). \tag{1.9}$$

For what follows it is important to note that the particle orbits, on the average, drift in the z direction. The velocity of this drift can be found from the obvious condition that

$$\frac{d}{dt}\langle\varphi\rangle = 0. \tag{1.10}$$

This condition gives

$$\left\langle\frac{d\vartheta}{dt}\right\rangle = \alpha\left\langle\frac{dz}{dt}\right\rangle = -\frac{c}{rB_0}\frac{d\Phi}{dr} - \frac{3\mu B_0}{m_j\omega_{cj} r^2}\left[\frac{2E(\varkappa)}{K(\varkappa)} - 1\right]\varepsilon_h. \tag{1.11}$$

Here the angle brackets denote an average taken over an oscillation period for the trapped particles according to the rule

$$\langle A(\varphi)\rangle = \frac{1}{2K(\varkappa)}\int_0^{\varphi_0} \frac{A(\varphi)\, d\varphi}{\sqrt{\varkappa^2 - \sin^2({}^1/_2\varphi)}}. \tag{1.12}$$

For weakly toroidal conditions $\varepsilon_t \neq 0$, it is clear that one can assume that the rapid banana-form motion of the trapped particles between the magnetic stoppers is maintained, but the r coordinate of the banana as well as $\langle\vartheta\rangle$, will vary slowly. The equation describing this slow motion can be found averaging Eq. (1.4),

$$\left\langle\frac{dr}{dt}\right\rangle = \frac{dr}{dt} = -\varepsilon_t \frac{\mu B_0}{m_j \omega_{cj} r}\langle\sin\vartheta\rangle. \tag{1.13}$$

Within the approximation employed here $B_\vartheta/B_z \ll \alpha$ (i.e., neglecting the rotational transformation) there is no rapid motion with respect to ϑ and consequently $\langle\sin\vartheta\rangle = \sin\langle\vartheta\rangle$.

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*This cannot be done even close to the axis for particles in transit.

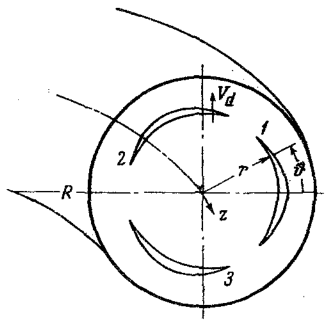

Fig. 1

It is clear from Eq. (1.13) that as a result of the toroidal condition the banana will drift across the magnetic surface. In the process of particle oscillation between regions with a strong helical magnetic field the radial toroidal drift keeps the same sign until the particle falls into a toroidal drift region of the opposite sign, as a result of the slow azimuthal $\langle\vartheta\rangle$ motion. The slower the banana moves with respect to $\langle\vartheta\rangle$ (and consequently with respect to z), the greater its departure from the original magnetic surface. In the approximation where the toroidal condition is very slight the motion with respect to $\langle\vartheta\rangle$ is described by Eq. (1.11).

First of all, we consider the case in which the electric field $d\Phi/dr \equiv 0$. Then for $\varkappa^2 = \varkappa_0^2 = 0.83$ the drift velocity of the banana along the z-axis goes to zero (the right-hand side of Eq. (1.11) vanishes). In the process of the slow drift motion such bananas turn out to be trapped within the limits of the bounded sections of the force lines and have anomalously large departures from the magnetic surface. We expand the expression for the drift velocity $d\langle\vartheta\rangle/dt$ in terms of a small departure from the point (x_0, r_0),

$$\frac{d\langle\vartheta\rangle}{dt} = -\frac{3\mu B_0\varepsilon_h}{m_j\omega_{cj}r_0K(\varkappa_0)}\frac{d}{d\varkappa_0^2}\left[2E(\varkappa_0)-K(\varkappa_0)\right]\left[\varkappa^2-\varkappa_0^2+(r-r_0)\frac{d\varkappa^2}{dr}\right]. \tag{1.14}$$

In terms of the variables

$$\tau = \frac{\mu B_0}{m_j\omega_{cj}r_0^2}t, \qquad x = \frac{r-r_0}{r_0}, \tag{1.15}$$

the system of the equations of motion (1.11) and (1.14) for the drift orbits (bananas) have the simple form

$$\dot{x}_\tau = -\varepsilon_t\sin\vartheta,\ \dot{\vartheta}_\tau = 5.4\varepsilon_h(\varkappa^2-\varkappa_0^2) - 5.4\varepsilon_h x. \tag{1.16}$$

Solving system (1.16) for the initial condition $x(\vartheta = \pi) = 0$, we find that the following quantity is conserved in the drift process:

$$I = x + \frac{\sigma}{2}\frac{\sqrt{\varepsilon_t}}{\sqrt{1.35\,\varepsilon_h}}\sqrt{2\eta^2-(1+\cos\vartheta)}, \tag{1.17}$$

and the variation of $\langle\vartheta\rangle$ is described by the equation

$$r\frac{d\langle\vartheta\rangle}{dt} = \frac{3\mu B_0}{m_j\omega_{cj}r}\sigma\sqrt{1.2\,\varepsilon_h\varepsilon_t\left[2\eta^2-(1+\cos\vartheta)\right]}, \tag{1.18}$$

$$2\eta^2 = 2.7\frac{\varepsilon_h}{\varepsilon_t}\left[\varkappa^2-\varkappa_0^2\right]^2. \tag{1.19}$$

It is clear from this that the drift orbits of particles with the parameter $0 < \eta^2 \leq 1$ precess within limits of the bounded sections of the lines of force with a period

$$T_{tj} = \frac{2m_j\omega_{cj}r^2K(\eta)}{\mu B_0\sqrt{2.7\,\varepsilon_t\,\varepsilon_h}}. \tag{1.20}$$

A graph of the maximum departure $(r - r_0)$ from the magnetic surface as a function of $\varkappa^2$ is given in Fig. 2. Bananas at the capture limit $\eta^2 = 1$ have the greatest displacement

$$\Delta r_t \Big|_{\eta^2=1} = \left(\frac{\varepsilon_t}{1.35\,\varepsilon_h}\right)^{1/2} r_0 \,. \tag{1.21}$$

Drift orbits which go right around the whole toroid in the drift process (we refer to these as transit bananas) depart considerably less from the magnetic surface,

$$\Delta r_u \Big|_{\varkappa^2=0.1} \approx \frac{2\varepsilon_t}{3\varepsilon_h} r \,. \tag{1.22}$$

The precession period of such bananas is equal to

$$T_{uj} = \frac{2\pi m_j \omega_{cj} r^2}{3\mu B_0 \varepsilon_h} \left[\frac{2E(\varkappa)}{K(\varkappa)} - 1\right]^{-1} \tag{1.23}$$

In the presence of a fairly strong electric field

$$-\frac{c}{B_0}\frac{d\Phi}{dr} \equiv v_E \gg \frac{\mu B_0 \varepsilon_h}{m_j \omega_{cj} r} \tag{1.24}$$

the drift velocity $\langle\vartheta\rangle$ does not vanish anywhere. As a result of this the drift orbits move along the torus with almost constant velocity and gradually go around regions with different signs of toroidal drift. In this case trajectories confined within the limited region of the magnetic lines of force do not exist at all. Solving the equation

$$\frac{d\langle\vartheta\rangle}{dt} = -\frac{c}{B_0}\frac{d\Phi}{dr} \tag{1.25}$$

together with Eq. (1.13), we have

$$r(\langle\vartheta\rangle) - r_0 = \varepsilon_t \frac{\mu B_0}{m_j \omega_{cj} v_E}(\cos\langle\vartheta\rangle + 1), \quad T_E = \frac{r_0}{v_E} \quad \left(v_E \approx -\frac{c}{B_0}\frac{d\Phi}{dr}\right) \tag{1.26}$$

2. The banana kinetic equation. As a next step it is natural to write the kinetic equation describing the average motion of the bananas. This is obtained by averaging the ordinary drift kinetic equation over a period of trapped-particle rapid motion in the magnetic field. We introduce the distribution function $F_j(r, \langle\vartheta\rangle, \mu, \varkappa^2, t)$ for the bananas. It obeys the kinetic equation

$$\frac{\partial F_j}{\partial t} + \frac{d\langle\vartheta\rangle}{dt}\frac{\partial F_j}{\partial\vartheta} + \frac{dr}{dt}\frac{\partial F_j}{\partial r} + \frac{\partial\varkappa^2}{\partial r}\frac{dr}{dt}\frac{dF_j}{\partial\varkappa^2} = \langle S\{F_j\}\rangle \,, \tag{2.1}$$

where the drift equations (1.11) and (1.13) of the bananas must be used for $d\langle\vartheta\rangle/dt$ and dr/dt, while

$$\frac{\partial\varkappa^2}{\partial r} = \frac{\partial}{\partial r}\left[\frac{1}{4\pi\mu B_0 \varepsilon_h(r)}\left\{m_j\left(\frac{2}{m_j}[E - e_j\Phi(r) - \mu B_0(1 - \varepsilon_h(r))]\right)^{1/2} - \right.\right.$$
$$\left.\left. - \frac{c}{\alpha B_0 r}\frac{d\Phi}{dr}\right\}^2\right].$$

The element of phase space for the bananas after integration over a period $3(\vartheta - \alpha z)$ has the form

$$\frac{4\pi d\mu B_0}{m_j} d\,\frac{1}{2\pi}\oint v_\parallel\, dt = 4\pi\frac{d\mu B_0}{m_j} d\oint\left(2\varepsilon_h\frac{\mu B_0}{m_j}[2\varkappa^2 - (1 - \cos t)]\right)^{1/2}\frac{dt}{2\pi} =$$

$$= 8\sqrt{\varepsilon_h}\left(\frac{\mu B_0}{m_j}\right)^{1/2} d\,\frac{\mu B_0}{m_j} K(\varkappa)\, d\varkappa^2 \qquad (\varkappa^2 \ll 1)\,. \tag{2.2}$$

The collision integral in (2.1) for the bananas can be obtained from the following considerations. For $\varepsilon_h \ll 1$ the number of bananas is small compared with the transit particles. Consequently, the collision integral can be linearized if we neglect the collisions between the bananas. We begin from the familiar expression [8] for the linearized collision integral:

$$S\{F_j\} = \sum_{j'}\frac{2\pi\lambda e_j^2 e_{j'}^2}{m_j}\frac{\partial}{\partial v_\alpha}\left\{\left(\eta_{j'} + \eta_{j'}' - \frac{\eta_{j'}}{2x_{j'}}\right)\left[\frac{\delta_{\alpha\beta}}{v} - \frac{v_\alpha v_\beta}{v^3}\right] + \right.$$

$$+\frac{v_\alpha v_\beta}{v^3}\frac{\eta_{j'}}{x_{j'}}\Big\}\left(\frac{\partial F_j}{m_j \partial v_\beta}+\frac{2v_\beta}{m_{j'}v_{Tj'}^2}F_j\right),$$

$$\eta_j \equiv \eta(x_j) = \frac{2}{\sqrt{\pi}}\int\limits_0^{x_j} e^{-t}\sqrt{t}\,dt, \quad \eta' = \frac{\partial \eta(x)}{\partial x}, \quad x_j = \frac{v^2}{v_{Tj}^2}. \tag{2.3}$$

This expression can be still further simplified if we take into account that the distribution of trapped particles is most sensitive to changes in the longitudinal velocity and so all the remaining derivatives can be neglected in Eq. (1.3). Moreover passing to the new variables μ, $\varkappa^2$, φ according to the relation which follows from Eq. (1.3) and (1.6), we have

$$v_\parallel = \frac{c}{\alpha B_0 r}\frac{d\Phi}{dr} + 2\sigma\left[\frac{\mu B_0}{m_j}\varepsilon_h\left(\varkappa^2 - \sin^2\frac{\varphi}{2}\right)\right]^{1/2}. \tag{2.4}$$

Rewriting (2.3) in the form [1]

$$S\{F_j\} = \frac{\nu_j}{\varepsilon_h}A(x_j)\,\sigma\sqrt{\varkappa^2-\sin^2 {}^1\!/_2\varphi}\,\frac{\partial}{\partial\varkappa^2}\Big\{\sigma\sqrt{\varkappa^2-\sin^2 {}^1\!/_2\varphi}\;\times$$
$$\times\left(\frac{\partial F_j}{\partial\varkappa^2}+2x_j\varepsilon_h F_j\right)+c_j\sqrt{2x_j\varepsilon_h}\,F_j\Big\}. \tag{2.5}$$

Here

$$\nu_j = \frac{16\sqrt{\pi}\,\lambda e^4 n}{3\sqrt{m_j}\,v_{Tj}^3}, \qquad A(x_j) = \frac{3\sqrt{\pi}}{4}\sum_{j'}\left(\eta_{j'}+\eta_{j'}'-\frac{\eta_{j'}}{2x_{j'}}\right)x_j^{-3/2},$$
$$c_j = \frac{c}{\alpha B_0 r v_{Tj}}\frac{d\Phi}{dr}.$$

Finally, to obtain the collision integral for the bananas this expression must be averaged over a period of the rapid oscillation of the trapped particles in the magnetic field according to rule (1.12). As a result we have

$$\langle S\{F_j\}\rangle = \frac{\nu_j}{\varepsilon_h}A(x_j)\frac{1}{K(\varkappa)}\frac{\partial}{\partial\varkappa^2}\int\limits_0^{\varkappa^2} K(t^{1/2})\,dt\left(\frac{\partial F_j}{\partial\varkappa^2}+2x_j\varepsilon_h F_j\right). \tag{2.6}$$

It should be kept in mind that the banana kinetic equation (3.1) is valid only when a trapped particle undergoes no collisions at all during a period of the rapid motion:

$$\frac{\nu_j}{\varepsilon_h} \ll \frac{1}{\tau_j} \approx \alpha v_{Tj}\sqrt{\varepsilon_h}. \tag{2.7}$$

otherwise the very concept of the bananas does not have any meaning.

3. Plasma transport coefficients in the absence of an electric field. In the final analysis the electric field $\Phi(r)$ must be determined from the condition of ambipolar diffusion. In what follows we consider the special limiting case in which, as will be clear, the diffusion is ambipolar for $d\Phi/dr = 0$. Since the particles in this case depart strongly from the magnetic surfaces we can expect a large diffusion here.

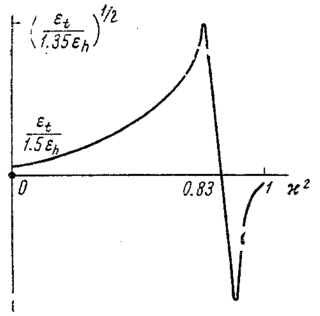

Fig. 2

We consider the situation in which the collision frequency of the trapped particles lies in an interval between the inverse precession periods of trapped and transit bananas:

$$T_{uj}^{-1} \gg \frac{\nu_j}{\varepsilon_h} \gg \frac{\varepsilon_t}{\varepsilon_h} T_{tj}^{-1} . \tag{3.1}$$

Then in a small region of "energy" $\varkappa^2$ of the trapped bananas the collisions manage to establish a Maxwell distribution and so the complete solution of the banana kinetic equation (2.1) can be sought in the form of an expansion in the small parameter $(\varepsilon_t/\varepsilon_h) \ll 1$. To do this we write the distribution function F_j in the form of a Maxwell function $F_j^{(0)}$ together with a small increment $F_j^{(1)}$ since the system is toroidal:

$$F_j = F_j^{(0)} (r, E) + F_j^{(1)} (r, \vartheta, \mu, \varkappa^2),$$
$$F_j^{(0)} (r, E) = \frac{n_{0j}(r)}{\pi^{3/2} v_{Tj}^3} \exp\left\{-\frac{2E}{m_j v_{Tj}^2}\right\} . \tag{3.2}$$

In Eq. (2.1) for $F_j^{(1)}$ we confine ourselves to the approximation for the collision integral and insert explicit expressions for the drift velocities of the bananas from (1.11) and (1.13). The resulting equation

$$-\frac{3\mu B_0}{m_j \omega_{cj} r^2} \varepsilon_h \left(\frac{2E(\varkappa)}{K(\varkappa)} - 1\right) \frac{\partial F_j^{(1)}}{\partial \vartheta} + \frac{\nu_j}{\varepsilon_h} F_j^{(1)} = \frac{\mu B_0}{m_j \omega_{cj} r} \sin\vartheta \frac{\partial F_j^{(0)}}{\partial r} \tag{3.3}$$

has the following solution:

$$F_j^{(1)} = -\frac{\varepsilon_t}{3\varepsilon_h} \frac{\partial F_j^{(0)}}{\partial \ln r} \left\{P \frac{\cos\vartheta}{2E(\varkappa)/K(\varkappa) - 1} - \pi \sin\vartheta\,\delta\left(\frac{2E(\varkappa)}{K(\varkappa)} - 1\right)\right\} . \tag{3.4}$$

Here the symbol P is used to denote the principal value of the singular expression.

Multiplying function (3.4) by the drift velocity dr/dt of a banana and integrating (2.2) with respect to the phase volume of the bananas, we find the plasma flux across the magnetic field:

$$\langle n v_{rj}{}^{\vartheta} \rangle = -8\varepsilon_h^{1/2} \int_0^{2\pi} \frac{d\vartheta}{2\pi} \int_0^{\infty} \left(\frac{\mu B_0}{m_j}\right)^{1/2} d\,\frac{\mu B_0}{m_j} \int_0^1 d\varkappa^2 K(\varkappa) ,$$
$$F_j^{(1)} \frac{\mu B_0 \varepsilon_t}{m_j \omega_{cj} r} \sin\vartheta = -\frac{\varepsilon_t^2}{4.3\,\varepsilon_h^{1/2}} \frac{v_{Tj}^2}{\omega_{cj}} \frac{dn(r)}{dr} . \tag{3.5}$$

It follows from this that if inequalities (3.1) for the collision frequency of ions and electrons are satisfied (which, by the way, cannot always be satisfied simultaneously for both types of particles) the diffusion of particles in an isothermic plasma will be ambipolar.

If the collision frequency passes outside the limits of inequalities (3.1), the requirement that diffusion in a dense plasma be ambipolar imposes a restriction of the following form on the electric field strength:

$$e_j n \frac{d\Phi(r)}{dr} = -m_j v_{Tj}^2 \frac{dn(r)}{dr} . \tag{3.6}$$

Here the subscript j refers to the class of particles which diffuse most strongly in the absence of an electric field.

4. The effect of an unperturbed electric field on the transport process. An electric field with amplitude (3.6) reduces considerably the amount by which orbits depart from the magnetic surface, and also markedly smoothes the way this depends on the energy of the particle. It is thus natural to expect a marked decrease of the transport coefficients in a rarefied plasma.

First of all, we consider the case in which collisions are not very rare, when the drift orbits of the particles do not manage to perform complete precession periods over the trajectories (1.26) during the collision time, i.e.,

$$\frac{\nu_j}{\varepsilon_h} > T_E^{-1} . \tag{4.1}$$

The characteristic mixing scale is then the mean free path of a drift orbit between collisions:

$$\lambda_j \sim \frac{\mu B_0 \varepsilon_t}{m_j \omega_{cj} r} \Big/ \frac{\nu_j}{\varepsilon_h} \quad . \tag{4.2}$$

As the collision frequency decreases, the mixing scale increases and the transport coefficients increase correspondingly. To calculate these numerically, we look for a solution of the kinetic equation (2.1) in the form of an expansion with respect to smallness of the mean free path in the form (3.2). When condition (4.1) is satisfied it is sufficient to take into account only collisions and toroidal drift in the kinetic equation (2.1). Further, because of the smallness of displacement of the transit particles from the magnetic surface, the correction to the Maxwell distribution function also turns out to be considerably less than for the bananas. However, the requirement that the distribution function should be continuous imposes the condition that the correction $F_j^{(1)}$ for the bananas should vanish at the boundary of their phase volume:

$$F_j^{(1)}\big|_{\varkappa^2=1-0} = 0 \ . \tag{4.3}$$

Using the boundary condition (4.3) we find a solution of the linearized equation (2.1):

$$F_j^{(1)} = -\frac{\mu B_0}{m_j} \frac{\varepsilon_t \varepsilon_h}{\omega_{cj} r \nu_j A(x_j)} (\varkappa^2 - 1) \frac{\partial F_j^{(0)}}{\partial r} \sin\vartheta \ . \tag{4.4}$$

An expression for the flux of particles diffusing across the magnetic field is found by substituting the obtained result into Eq. (3.5):

$$\langle n v_{rj}^{\vartheta} \rangle = \frac{\varepsilon_t^2 \varepsilon_h^{1/2}}{(2\pi)^{3/2}} \left(\frac{r_{cj} v_{Tj} \varepsilon_h}{\nu_j r^2} \right) \frac{v_{Tj}^2}{\omega_{cj}} \int_0^\infty e^{-x_j} \frac{x_j^{5/2} dx_j}{A(x_j)} \left(\frac{dn}{dr} - \frac{2 e_j \Phi}{m_j v_{Tj}^2} n \right),$$

$$\frac{\varepsilon_t}{\varepsilon_h} \tau_j^{-1} > \frac{\nu_j}{\varepsilon_h} > T_E^{-1} \ . \tag{4.5}$$

Here the left-hand inequality as regards the collision frequency is an expression of the fact that the precession of the bananas becomes important only in a fairly rarefied plasma, when the free path of toroidal drift (4.2) of a banana exceeds the width of a banana itself. Otherwise the plasma mixing in the process of the rapid motion of a trapped particle along a banana, considered previously in [1], makes a larger contribution to the transport coefficients than the weak precession of the banana (Fig. 3).

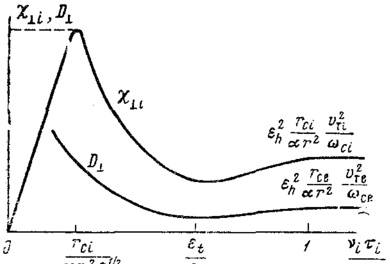

Fig. 3

It follows directly from Eq. (4.5) that the ion and electron flux across the magnetic field will be ambipolar only on condition that the electric field almost completely compensates the ion pressure (the case j = i in formula (3.6)).

The rate of ambipolar diffusion is then determined by the electrons, while the coefficients of ion thermal diffusivity $\chi_{\perp i}$ remains considerably greater than for the electrons, and can easily be found by the same method as the particle flux. A maximum

$$\chi_{\perp i} \sim \varepsilon_h^{1/2} \varepsilon_t^2 \frac{v_{Ti}^2}{\omega_{ci}}$$

is attained for a collision frequency on the order of one precession period of an orbit, and subsequently begins to decrease linearly with ν (see Fig. 3). The resulting expression (1.26) for the mixing length in this case enables us to estimate the quantity $\chi_{\perp i}$ without calculations:

$$\chi_{\perp i} \simeq \varepsilon_h^{1/2}\left(\frac{\nu_i}{\varepsilon_h}\right)(\Delta r)^2 = \nu_i \varepsilon_t^2 \varepsilon_h^{-1/2}\left(\frac{d \ln n_0}{dr}\right)^{-2}, \quad \frac{\nu_i}{\varepsilon_h} \ll T_E^{-1}. \tag{4.6}$$

As the collision frequency decreases further, the electric field changes sign and the electrons begin to make the greatest contribution to the thermal conductivity.

5. The motion of trapped particles in a tight stellarator. We now consider the case which is just the opposite of that considered previously. This is the case of a very tight stellarator.

We will use the magnetic field of a two-track stellarator close to the axis [5] as the model:

$$\begin{gathered}\mathbf{B} = B_0\{1 - \varepsilon_t \cos\vartheta\}\mathbf{e}_z + \alpha r b\{\cos 2(\vartheta - \alpha z) - {}^1/_2 b / B_0\}\mathbf{e}_\vartheta + \\ + \alpha r b \sin 2(\vartheta - \alpha z)\mathbf{e}_r \\ \varepsilon_t \gg \left(\frac{b}{B_0}\right)^2(\alpha r)^2, \quad \frac{b}{B_0} \ll 1.\end{gathered} \tag{5.1}$$

Two types of trapped particles must be distinguished. Some of these have a fairly large longitudinal velocity and pass freely across the local magnetic bottles, which arise as a result of the rapid oscillations of the line of force around its average position. Thus in the process of motion of such particles an averaging takes place over the period of the helical field and this case reduces to the case already considered, i.e., an axisymmetric field with an average rotatory transformation [1].

However, there also exists a group of particles with a very small longitudinal velocity

$$v_\parallel \lesssim v_\perp\left(\frac{b\varepsilon_t}{2B_0}\right)^{1/2} \tag{5.2}$$

and these particles turn out to be trapped in the limits of one period of the helical field. The cause of this is the fact that a line of force intersects the surface of the constant magnetic field where a maximum value of the field is obtained for a given portion of a line of force, twice during one period ((see Fig. 4); the trajectory is represented by the dashed line, and its projection onto the z = const plane is represented by the solid line). Just as in Section 2, we break up the motion into rapid oscillations along the line of force and drift in the toroidal magnetic field. Solving the equation for drift, averaged with respect to the rapid oscillations, we obtain the particle trajectory in the absence of an electric field:

$$\langle r(\langle\vartheta\rangle)\rangle = \frac{r_0}{\cos\langle\vartheta\rangle + (b/B_0)(E(\varkappa)/K(\varkappa) - 0.5)}, \tag{5.3}$$

$$\varkappa^2 = \frac{E - \mu B_0(1 - \langle\varepsilon_t\rangle\cos\langle\vartheta\rangle - ({}^1/_2 b/B_0)\langle\varepsilon_t\rangle)}{\mu b\langle\varepsilon_t\rangle}. \tag{5.4}$$

We find from (5.3) and (5.4) that a particle drifts approximately in the plane of the constant magnetic field until it escapes, owing to a change in the parameter x^2. It is true that during this time it drifts a distance on the order of the dimensions of the system. In the presence of an equilibrium electric field (40) the toroidal drift of trapped particles is averaged with respect to the angle ϑ because of the rapid drift (see Section 2) and the trajectory coincides with that found previously (1.26). The departure of the particles from the magnetic surface in this case is reduced considerably in comparison with the case in which there is no electric field.

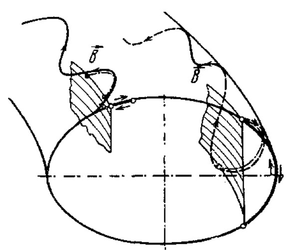

Fig. 4

The transport coefficients can be calculated in the same way as in Section 5. Trapped particles of the first type make the fundamental contribution to the transport processes when collisions are frequent, and particles of the second type, when they are rare. As before, the coefficients $D_\perp$ and $\chi_{\perp i}$ as functions of the collision frequency are described qualitatively in Fig. 3.

To obtain quantitative estimates we should keep in mind that the number of particles of the second type is considerably smaller and the collision frequency much greater than for particles of the first type.

$$\frac{\delta n}{n_0} \approx \frac{2}{\pi}\left(\frac{b\varepsilon_t}{B_0}\right)^{1/2}, \qquad \nu_{eff} \simeq \nu_j \bigg/ \left(\frac{b\varepsilon_t}{B_0}\right). \tag{5.5}$$

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March 1968